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Coherent Structures of Non-Identical Components

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COHERENT STRUCTURES
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1. Introduction

In [3] Moore and Shannon obtain some basic results concerning the reliability of two-terminal networks composed of independent components of equal reliability. In particular, they show that the reliability of the network plotted as a function of the component reliability is S-shaped, i.e. crosses the diagonal at most once and always from below. In [1] Birnbaum, Esary, and Saunders generalize the results of Moore and Shannon to what they call coherent structures; a coherent structure being, roughly, one whose performance does not deteriorate when failed components are replaced by functioning ones. Coherent structures include two-terminal networks, "k out of n" structures (structures which function if and only if at least k out of n components function), and many others. In [1] it is assumed, just as in [3], that components are independent and of identical reliability.

In the present paper we shall exploit a basic theorem on the covariance of increasing functions of random variables which permits us to discuss the case of coherent structures whose components are independent, but of differing reliabilities. We obtain first some convenient bounds on the reliability of structures, then a generalization of some statistical properties obtained in [1] for coherent structures, and finally a generalization of a differential inequality introduced in [3] which relates structural and component reliabilities. One of the consequences of the present approach is a very simple and direct demonstration of the S-shapedness results presented in [3] and [1].

2. Background and notation

The systems (or structures) we consider are capable of only two states of performance; either they function or they fail to function. Similarly the components from which the systems are built may function or fail to function. We associate with the i^{th} component of a system a binary variable x_i with $x_i = 1$ when the component functions and $x_i = 0$ when the component fails. The state of the entire set of components of the system is indicated by the vector $\underline{x} = (x_1, x_2, \ldots, x_n)$. The number of components n is called the order of the system. It is assumed that the state of the system is determined by the states of the components, so that the state of the system may be indicated by a binary function $\phi(\underline{x})$ with $\phi(\underline{x}) = 1$ when the system functions and $\phi(\underline{x}) = 0$ when the system fails; ϕ is called a structure function.

Within the class of binary systems of binary components we are particularly interested in those that are coherent. A system having structure function ϕ is coherent if:

(a) $\phi(\underline{x}) \geq \phi(\underline{y})$ whenever $\underline{x} \geq \underline{y}$, where by $\underline{x} \geq \underline{y}$ we mean $x_i \geq y_i$, $i = 1,2,\ldots,n$

(b) $\phi(\underline{1}) = 1$, where $\underline{1} = (1,1,\ldots,1)$

(c) $\phi(\underline{0}) = 0$, where $\underline{0} = (0,0,\ldots,0)$.

Functions $f(\underline{x})$ (binary or otherwise) which satisfy property (a) will be called increasing. A structure function ϕ which is increasing has been called [1] semi-coherent. It is immediate that the only semi-coherent systems which are not coherent are the two trivial cases $\phi(\underline{x}) \equiv 1$ and $\phi(\underline{x}) \equiv 0$. The increasing property of coherent systems seems descriptive of many real systems -- if sufficient components are functioning to cause the system to function, then the functioning of additional components can only improve matters; if sufficient components have failed to cause system failure, then the failure of additional components can only make matters worse.

In considering the reliability of systems we will suppose that each component functions with a certain probability. This is equivalent to associating to the i^{th} component a binary random variable X_i, where $p_i = \Pr[X_i = 1] = E[X_i]$ is the reliability of the component, and $q_i = 1 - p_i = \Pr[X_i = 0]$ is component "unreliability". The reliability $h(\underline{p})$ of the system is then

$$h(\underline{p}) = \Pr[\phi(\underline{X}) = 1 | \underline{p}] = E[\phi(X) | \underline{p}],$$

where $\underline{p} = (p_1, p_2, \ldots, p_n)$.

A coordinate x_i of a vector of binary variables $\underline{x}$ is inessential to a function $f(\underline{x})$ if

$$f(1_i, \underline{x}) = f(0_i, \underline{x})$$

for all vectors $(\cdot_i, \underline{x})$, where

$$(1_i, \underline{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

$$(0_i, \underline{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

$$(\cdot_i, \underline{x}) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n).$$

When the function is the structure function of a system an inessential coordinate corresponds to a component whose functioning or failure does not affect the performance of the system. Any coordinate which is not inessential to f is called <u>essential</u> to f.

A vector $\underline{x}$ for which a structure function $\phi(\underline{x}) = 1$ is a <u>path</u> of the system represented by ϕ. If $\phi(\underline{x}) = 0$, $\underline{x}$ is a <u>cut</u>. When the system is coherent, the coordinates of a path which are one indicate a set of components which by functioning are sufficient to cause the system to function; the coordinates of a cut which are zero indicate a set of components which by failing are sufficient to cause the system to fail.

3. Covariance of increasing functions

A key tool in our analysis of structural reliability is a specialized version of the Tchebichev inequality [2] stating that the covariance of two increasing functions of independent binary random variables is non-negative. A proof adapted to this particular case is not readily accessible in the literature. One is given here that is an immediate consequence of a useful representation for the covariance of two functions of independent binary random variables.

If $x_1, x_2, \ldots, x_n$ are binary variables, then for any function $f(\underline{x})$ the representation

$$f(\underline{x}) = x_i f(1_i, \underline{x}) + (1 - x_i) f(0_i, \underline{x}) \tag{3.1}$$

holds for each $i = 1,2,\ldots,n$. Let $X_1, X_2, \ldots, X_n$ be independent binary random variables. Since X_i is independent of $f(1_i, \underline{X})$ and $f(0_i, \underline{X})$ the representation

$$Ef(\underline{X}) = p_i E[f(1_i, \underline{X})] + q_i E[f(0_i, \underline{X})] \tag{3.2}$$

then follows from (3.1).

Lemma 3.1

Let $X_1, X_2, \ldots, X_n$ be independent binary random variables. For any functions $f_j(\underline{X})$, $j = 1,2$, and any X_i

$$\begin{aligned} \operatorname{cov}[f_1(\underline{X}), f_2(\underline{X})] = {} & p_i \cdot \operatorname{cov}[f_1(1_i, \underline{X}), f_2(1_i, \underline{X})] \\ & + q_i \cdot \operatorname{cov}[f_1(0_i, \underline{X}), f_2(0_i, \underline{X})] \\ & + p_i q_i \cdot E[f_1(1_i, \underline{X}) - f_1(0_i, \underline{X})] \cdot E[f_2(1_i, \underline{X}) - f_2(0_i, \underline{X})]. \end{aligned}$$

Proof

From (3.2)

$$E[f_1(\underline{X})f_2(\underline{X})] = p_i \cdot E[f_1(1_i,\underline{X})f_2(1_i,\underline{X})] + q_i \cdot E[f_1(0_i,\underline{X})f_2(0_i,\underline{X})]$$

$$E[f_j(\underline{X})] \qquad = p_i \cdot E[f_j(1_i,\underline{X})] + q_i \cdot E[f_j(0_i,\underline{X})], \quad j = 1,2$$

Using these expansions the representation of the lemma is easily checked when each covariance involved is written in the form

$$\operatorname{cov}[f_1,f_2] = E(f_1f_2) - E(f_1) \cdot E(f_2).$$

Theorem 3.1

Let $X_1,X_2,\ldots,X_n$ be independent binary random variables. Let $f_j(\underline{X})$, $j = 1,2$, be increasing functions. Then

$$\operatorname{cov}[f_1(\underline{X}),f_2(\underline{X})] \geq 0.$$

Proof

We proceed by an induction on the order n of the functions.

For $n = 1$, from Lemma (3.1),

$$\operatorname{cov}[f_1(X_1),f_2(X_1)] = p_1q_1 \cdot [f_1(1) - f_1(0)] \cdot [f_2(1) - f_2(0)],$$

an expression which is clearly non-negative for increasing f_1 and f_2.

Assume that the covariance of any two increasing functions of order $n - 1$ is non-negative. If $f_j(\underline{X})$, $j = 1,2$, are increasing and of order n, then the related functions $f_j(1_i,\underline{X})$, $f_j(0_i,\underline{X})$, $j = 1,2$ are all increasing and of order $n - 1$. Thus the first two terms of the representation of Lemma 3.1 are non-negative. Since $f_j(1_i,\underline{X}) \geq f_j(0_i,\underline{X})$, $j = 1,2$, the third term of the same representation is also non-negative.

Corollary

Under the hypotheses of Theorem 3.1 and the additional assumption that $0 < p_i < 1$ for each $i = 1,2,\ldots,n$, a necessary and sufficient condition for

$$\operatorname{cov}[f_i(\underline{X}), f_2(\underline{X})] > 0$$

is that some variable x_i be essential to both f_1 and f_2.

Proof

If no x_i is essential to both f_1 and f_2, then $f_1(\underline{X})$ and $f_2(\underline{X})$ are themselves independent random variables and their covariance is zero.

On the other hand suppose some x_i is essential to both f_1 and f_2. From Lemma 3.1 $\operatorname{cov}[f_1(\underline{X}), f_2(\underline{X})]$ can be expanded about this particular variable. We shall show that the third term of this expansion is strictly positive. That x_1 is essential to both functions means that there exists vectors $(\cdot_i, \underline{x}^{(j)})$, $j = 1,2$, such that

$$f_j(1_i, \underline{x}^{(j)}) > f_j(0_i, \underline{x}^{(j)}).$$

Then, letting $P(\cdot_i, \underline{x}) = \prod_{k \neq i} P[X_k = x_k]$,

$$\begin{aligned} E[f_j(1_i, \underline{X}) - f_j(0_i, \underline{X})] &= \sum_{\underline{x}} [f_j(1_i, \underline{x}) - f_j(0_i, \underline{x})]\, P(\cdot_i, \underline{x}) \\ &\geq [f_j(1_i, \underline{x}^{(j)}) - f_j(0_i, \underline{x}^{(j)}]\, P(\cdot_i, \underline{x}^{(j)}) > 0, \end{aligned}$$

for $j = 1,2$. Since also $p_i q_i > 0$, the third term is indeed strictly positive.

4. Approximations to reliability

We shall develop some convenient upper and lower bounds on the reliability of coherent systems whose components are independent but not necessarily of the same reliability. The following three lemmas will be useful.

Lemma 4.1

Let $X_1, X_2, \ldots, X_n$ be independent binary random variables. Define $f_j(\underline{X}) = \prod_{i \in A_j} X_i$, where A_j is a subset of $1,2,\ldots,n$, $j = 1,2,\ldots,m$.

Then

(4.1) $P[f_1 = 0,\ldots,f_m = 0] \geq P[f_1 = 0,\ldots,f_r = 0] \cdot P[f_{r+1} = 0,\ldots,f_m = 0]$

for $r = 1,2,\ldots,m$.

Proof

Define

$$F_1 = \begin{cases} 0 & \text{if each } f_j = 0,\ j = 1,2,\ldots,r \\ 1 & \text{otherwise} \end{cases}$$

$$F_2 \quad \begin{cases} 0 & \text{if each } f_j = 0,\ j = r+1,r+2,\ldots,m \\ 1 & \text{otherwise} \end{cases}$$

Thus $F_1 = 1 - \prod_{j=1}^{r} (1 - f_j)$, $F_2 = 1 - \prod_{j=r+1}^{m} (1 - f_j)$. Since each f_j is increasing, F_1 and F_2 are increasing. By Theorem 3.1 $E(F_1F_2) \geq E(F_1)\cdot E(F_2)$ or equivalently $E(1 - F_1)(1 - F_2) \geq E(1 - F_1)\cdot E(1 - F_2)$

or equivalently $P[F_1 = 0,\ F_2 = 0] \geq P[F_1 = 0] \cdot P[F_2 = 0]$.

Repeated application of Lemma 4.1 yields

Lemma 4.2

Under the same hypothesis as in Lemma 4.1,

(4.2) $$P[f_1 = 0_1, \ldots, f_m = 0] \geq \prod_{j=1}^{m} P[f_j = 0].$$

Lemma 4.3

Let $x_1, x_2, \ldots, x_n$ be independent binary variables. Define $f_j(\underline{x}) = \prod_{i \in A_j} x_i$, where A_j is a subset of $\{1,2,\ldots,n\}$, $j = 1,2,\ldots,r$. A sufficient condition that each x_i such that $i \in A_j$ for some j be essential to

$$F = 1 - \prod_{j=1}^{r} (1 - f_j)$$

is that no one of the sets A_j should be wholly included in any other.

Proof

Consider some x_{i_0}, $i_0 \in A_{j_0}$. For $i \neq i_0$ let

$$x_i^{(0)} = \begin{cases} 1 & \text{if } i \in A_{j_0} \\ 0 & \text{otherwise} \end{cases}$$

For $j \neq j_0$, $A_j - A_{j_0}$ is not empty so that $f_j(x_{i_0}, \underline{x}^{(0)}) = 0$. Thus $F(x_{i_0}, \underline{x}^{(0)}) = 1 - f_{j_0}(x_{i_0}, \underline{x}^{(0)}) = 1 - x_{i_0}$. Then $F(1_{i_0}, \underline{x}^0) = 1$, $F(0_{i_0}, \underline{x}^0) = 0$

which shows x_{i_0} to be essential to F.

<u>Remark</u>

If $A_1, A_2, \ldots, A_m$ as considered in the hypothesis of Lemmas 4.1 and 4.2 satisfy the condition that no one of them is a subset of another and if there is at least one x_i such that i is an element of at least two of the sets A_j, then with respect to Lemma 4.1 there is an r for which x_i is essential to the functions F_1 and F_2 defined in the proof. Then the inequality obtained in Lemma 4.1 is a strict inequality for that r, and consequently the inequality obtained in Lemma 4.2 is also strict.

Every coherent system has a finite number of minimal paths, i.e. vectors $\underline{z}$ for which $\phi(\underline{z}) = 1$ and such that if $\underline{x} < \underline{z}$, then $\phi(\underline{x}) = 0$. From a more physical point of view the elements of $\underline{z}$ which are 1's correspond to a smallest set of components which by functioning cause the system to function. Let us call the set of components indicated by the unit elements of a minimal path a <u>minimal path set</u>. Since the system functions if, and only if, all the components in at least one of the minimal path sets function, a representation of the system is obtained by imagining that the components of each minimal path set act in series and that the minimal path subsystems so obtained act in parallel. In such a representation of the system the same component may occur in more than one minimal path set making it necessary to suppose some deus-ex-machina which causes all replications of the same component to function or fail simultaneously. It is plausible that if in the representation each replica of the same component were replaced by an independently operating component of the same reliability, there would be an increased chance that the components of some minimal path set would all function. Thus a computation of reliability which treats the system as a set of indepenaent minimal paths acting in parallel should furnish an upper bound on actual system reliability.

Similarly every coherent system has a finite number of minimal cuts, i.e. vectors $\underline{y}$ for which $\phi(\underline{y}) = 0$ and such that if $\underline{x} > \underline{y}$, then $\phi(\underline{x}) = 1$. The set of components corresponding to the elements of a minimal cut which are 0's is a smallest set of components which by all failing cause the structure to fail; we call such a set a minimal cut set. Since the system functions unless all the components of some minimal cut set fail, it can be represented as one in which the components of each minimal cut set act in parallel and the minimal cut subsystems so obtained act in series. In this representation it is plausible that if the components which occur in more than one minimal cut were replaced in each occurrence by independently operating components of the same reliability, there would be an increased chance for all components of some minimal cut to fail, and that a computation of reliability which treats the system as a set of independently operating minimal cuts acting in series would give a lower bound on actual system reliability.

Theorem 4.1

Let ϕ be the structure function of a coherent system with independent components, not necessarily of identical reliability.

Define

$$\alpha_j = \begin{cases} 1 & \text{if all components of the } j^{th} \text{ minimal path set function} \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1,2,\ldots,a$, where a is the number of minimal paths of ϕ, and

$$\beta_k = \begin{cases} 0 & \text{if all components of the } k^{th} \text{ minimal cut set fail} \\ 1 & \text{otherwise} \end{cases}$$

for $k = 1,2,\ldots,b$, where b is the number of minimal cuts of ϕ. Then

$$(4.3) \qquad \prod_{k=1}^{b} P[\beta_k = 1] \leq P[\phi = 1] \leq 1 - \prod_{j=1}^{a} (1 - P[\alpha_j = 1]).$$

<u>Proof</u>

Let $A_1, A_2, \ldots, A_a$ be the minimal path sets of the system. Then

$$\alpha_j(\underline{X}) = \prod_{i \in A_j} X_i, \quad j = 1,2,\ldots,a.$$

Since $\phi = 0$ if, and only if, $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_a = 0$ we have from Lemma 4.2

$$P[\phi = 0] \geq \prod_{j=1}^{a} P[\alpha_j = 0]$$

or equivalently

$$P[\phi = 1] \leq 1 - \prod_{j=1}^{a} (1 - P[\alpha_j = 1).$$

Let $B_1, B_2, \ldots, B_b$ be the minimal cut sets of the system. Then

$$\beta_k(\underline{X}) = 1 - \prod_{i \in B_k} (1 - X_i), \quad k = 1,2,\ldots,b.$$

Define $\theta_i = 1 - X_i$, $i = 1,2,\ldots,n$, $f_k = 1 - \beta_k$, and $F = 1 - \phi$. Then

$$f_k(\underline{X}) = \prod_{i \in B_k} \theta_i,$$

and since $\phi = 1$ if, and only if, $\beta_1 = 1, \beta_2 = 1, \ldots, \beta_b = 1$ or equivalently $F = 0$ if, and only if, $f_1 = 0, f_2 = 0, \ldots, f_b = 0$ we have

$$P[F = 0] \geq \prod_{k=1}^{b} P[f_k = 0]$$

or equivalently

$$P[\phi = 1] \geq \prod_{k=1}^{b} P[\beta_k = 1].$$

Remark

The minimal path sets $A_1, A_2, \ldots, A_a$ satisfy the condition that no one of them is a subset of another as a consequence of their definition. It follows that if there is any overlap between minimal path sets, i.e., the same component occurring in two or more sets, and if $0 < p_i < 1$, $i = 1,\ldots,n$, then the right hand side of (4.3) is a strict inequality. If there is no overlap, i.e. the minimal path sets are disjoint, then the functions $\alpha_1, \alpha_2, \ldots, \alpha_a$ are independent and equality is obtained. The same criterion applied to minimal cut sets distinguishes between strict inequality and equality on the left side of (4.3).

Example 4.1

The diagram in Figure 4.1 represents a coherent system having seven components.

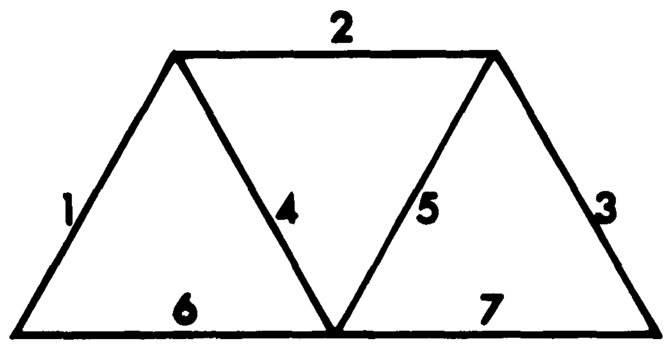


Figure 4.1

As may be readily determined by inspection, the system has the minimal path sets $\{6,7\}$, $\{1,2,3\}$, $\{1,4,7\}$, $\{3,5,6\}$, $\{1,3,4,5\}$, $\{1,2,5,7\}$, $\{2,3,4,6\}$, and the minimal cut sets $\{1,6\}$, $\{3,7\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{1,4,5,7\}$, $\{3,4,5,6\}$. From the remark preceding Theorem 4.1 (also see [1, Section 2.7.7]) the structure function of the system may be found by writing

$$\begin{array}{ll}
\alpha_1 = x_6x_7 & \beta_1 = 1 - (1-x_1)(1-x_6) \\
\alpha_2 = x_1x_2x_3 & \beta_2 = 1 - (1-x_3)(1-x_7) \\
\alpha_3 = x_1x_4x_7 & \beta_3 = 1 - (1-x_2)(1-x_4)(1-x_6) \\
\alpha_4 = x_3x_5x_6 & \beta_4 = 1 - (1-x_2)(1-x_5)(1-x_7) \\
\alpha_5 = x_1x_3x_4x_5 & \beta_5 = 1 - (1-x_1)(1-x_4)(1-x_5)(1-x_7) \\
\alpha_6 = x_1x_2x_5x_7 & \beta_6 = 1 - (1-x_3)(1-x_4)(1-x_5)(1-x_6) \\
\alpha_7 = x_2x_3x_4x_6 &
\end{array}$$

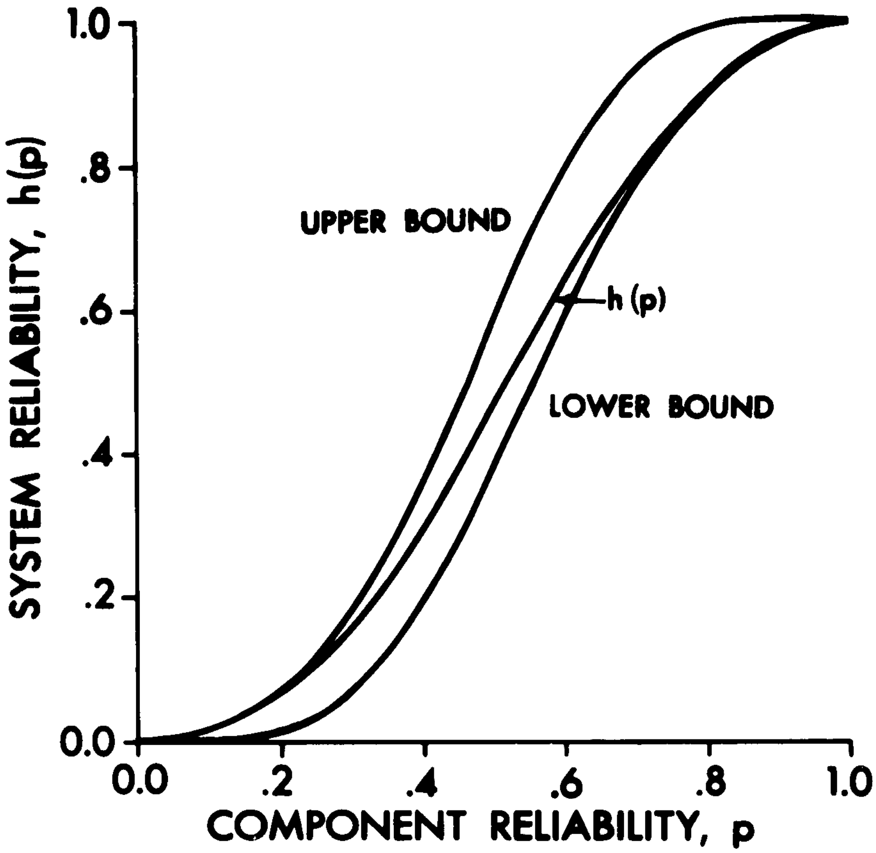
and using either of the representations

$$\phi = 1 - \prod_{j=1}^{7} (1 - \alpha_j) = \prod_{k=1}^{6} \beta_k. \tag{4.4}$$

The computation of the actual system reliability function by taking the expectation of ϕ in (4.4) is somewhat tedious. In this case the result is too long an expression to be given here. On the other hand the upper and lower bounds of (4.3) can be readily obtained since

$$\Pr[\alpha_j = 1] = \prod_{i \in A_j} p_i , \qquad \Pr[\beta_k = 1] = 1 - \prod_{i \in B_k} (1 - p_i) .$$

Figure 4.2 is a plot of the actual reliability function and the bounding functions in the case $p_i = p$, $i = 1,2,\ldots,n$, and furnishes an indication of the precision of the bounds.



BOUNDS ON SYSTEM RELIABILITY BASED ON MINIMAL PATHS AND MINIMAL CUTS FOR SEVEN-COMPONENT SYSTEM

Figure 4.2

5. Generalization of some properties of coherent structures to the case of components with non-identical reliabilities.

A number of interesting properties of coherent structures whose components have identical reliabilities were established in [1]. We shall prove some of these properties valid for coherent structures whose components do not have identical reliabilities and present some additional results for this case.

Theorem 5.1

Let $X_1, X_2, \ldots, X_n$ be independent binary random variables. Let $f(\underline{x})$ be increasing. Then

$$\text{cov}[f(\underline{X}), X_i] \geq 0, \quad i = 1,2,\ldots,n.$$

If in addition $0 < p_j < 1$, $j = 1,2,\ldots,n$ and x_i is essential to f, then

$$\text{cov}[f(\underline{X}), X_i] > 0.$$

Proof

Since the function of $\underline{x}$ identically equal to x_i is increasing the result follows from Theorem 3.1 and its corollary.

The representation of Lemma 3.1 gives in the present case

$$\text{cov}[f(\underline{X}), X_i] = p_i q_i \cdot E[f(1_i, \underline{X}) - f(0_i, \underline{X})]. \tag{5.1}$$

From (3.2)

$$(5.2) \qquad \frac{\partial h(\underline{p})}{\partial p_i} = \frac{\partial Ef(\underline{X})}{\partial p_i} = E[f(1_i,\underline{X}) - f(0_i,\underline{X})]$$

so that

$$(5.3) \qquad \text{cov}[f(\underline{X}),X_i] = p_i q_i \frac{\partial Ef(\underline{X})}{\partial p_i}.$$

Thus, assuming $0 < p_i < 1$, a necessary and sufficient condition that $\frac{\partial Ef}{\partial p_i} \geq 0$ at $\underline{p} = (p_1,p_2,\ldots,p_n)$ is that $\text{cov}[f,X_i] \geq 0$ at the same $\underline{p}$.

If ϕ is a structure function, then the function $h(\underline{p}) = E[\phi(\underline{X})|\underline{p}]$ is the structural reliability corresponding to component reliabilities $\underline{p}$. If ϕ represents a semi-coherent structure, ϕ is increasing, so that $\frac{\partial h}{\partial p_i} \geq 0$, $i = 1,2,\ldots,n$. The partial derivative of structural reliability with respect to a component reliability is identically zero if the component is not essential to the structure and, on the interior of the space of component reliabilities, strictly positive if the component is essential.

<u>Theorem 5.2</u>

Let $X_1,X_2,\ldots,X_n$ be independent binary random variables. Let $f(x)$ be increasing. Then

$$\text{cov}[f(\underline{X}),S(\underline{X})] \geq 0,$$

where $S(\underline{x}) = \sum_{i=1}^{n} x_i$.

Proof

Since

$$\mathrm{cov}[f,S] = \sum_{i=1}^{n} \mathrm{cov}[f,X_i]$$

the result follows immediately from Theorem 5.1.

We define a function $f(\underline{X})$ of n binary random variables $X_1,X_2,\ldots,X_n$ to be <u>increasing in expectation</u> if

$$E[f(\underline{X})|S(\underline{X}) = k+1] \geq E[f(\underline{X})|S(\underline{X}) = k], \quad k = 0,1,\ldots,n-1,$$

where $S(\underline{x}) = \sum_{i=1}^{n} x_i$.

This definition extends the definition given in [1] of a structure function ϕ <u>semi-coherent in probability</u> if

$$P[\phi = 1|S = k+1] \geq P[\phi = 1|S = k], \quad k = 0,1,\ldots,n-1.$$

Theorem 5.3

Let $X_1,X_2,\ldots,X_n$ be independent binary random variables. Let $f(\underline{x})$ be an increasing function. Then $f(\underline{X})$ is increasing in expectation.

Proof

We shall use induction to prove f increasing in expectation. For $n = 1$ we need $E(f|S = 1) \geq E(f|S = 0)$, or equivalently $f(1) \geq f(0)$.

Now assume that f increasing implies that f is increasing in expectation for functions of order $n - 1$. Define

$$T_i(\underline{X}) = \sum_{\substack{j=1 \\ j \neq i}}^{n} X_j .$$

Then

$$E[f(\underline{X})|S(\underline{X}) = k] = E[f(1_i,\underline{X})|T_i(\underline{X}) = k - 1] \cdot P[X_i = 1|S(\underline{X}) = k]$$

$$+ E[f(0_i,\underline{X})|T_i(\underline{X}) = k] \cdot P[X_i = 0|S(\underline{X}) = k].$$

Thus with the addition and subtraction of suitable terms, we may write

$$E[f|s = k + 1] - E[f|S = k]$$

$$= P[X_i = 1|S = k] \cdot \{E[f(1_i,\underline{X})|T_i = k] - E[f(1_i,\underline{X})|T_i = k -1]\}$$

$$+ P[X_i = 0|S = k + 1] \cdot \{E[f(0_i,\underline{X})|T_i = k + 1] - E[f(0_i,\underline{X})|T_i = k]\}$$

$$+ E[f(1_i,\underline{X})|S = k] \cdot \{P[X_i = 1|S = k + 1] - P[X_i = 1|S = k]\}$$

$$+ E[f(0_i,\underline{X})|S = k] \cdot \{P[X_i = 0|S = k + 1] - P[X_i = 0|S = k]\}$$

By inductive hypothesis

$$E[f(1_i,\underline{X})|T_i = k] \geq E[f(1_i,\underline{X})|T_i = k - 1]$$

and

$$E[f(0_i,\underline{X})|T_i = k + 1] \geq E[f(0_i,\underline{X})|T_i = k].$$

Also

$$P[X_i = 1|S = k + 1] - P[X_i = 1|S = k] = P[X_i = 0|S = k] - P[X_i = 0|S = k + 1] .$$

Thus we need only prove:

(a) $E[f(1_i,\underline{X})|S = k] \geq E[f(0_i,\underline{X})|S = k]$

(b) $P[X_i = 1|S = k + 1] \geq P[X_i = 1|S = k]$

To show (a) simply write

$$E[f(1_i,\underline{X})|S = k] - E[f(0_i,\underline{X})|S = k]$$

$$= E[f(1_i,\underline{X}) - f(0_i,\underline{X})|S = k] \geq 0,$$

since $f(1_i,\underline{x}) \geq f(0_i,\underline{x})$. Note that (b) is equivalent to

$$\frac{p_i P[T_i = k]}{P[S = k + 1]} \geq \frac{p_i P[T_i = k - 1]}{P[S = k]} .$$

Using the expansion

$$P[S = k] = p_i P[T_i = k - 1] + (1 - p_i)P[T_i = k]$$

(b) becomes equivalent to

$$\{P[T_i = k]\}^2 \geq P[T_i = k + 1] \cdot P[T_i = k - 1].$$

This last inequality holds since T_i, the convolution of binomial random variables, has a monotone likelihood ratio.

Theorem 5.3 contains the result that when components are independent a semi-coherent structure ϕ is semi-coherent in probability for any set of component reliabilities $p_1,p_2,\ldots,p_n$. That a coherent structure ϕ is <u>coherent in probability</u>, i.e. semi-coherent in probability and such that $P(\phi = 1|S = n) = 1$, $P(\phi = 0|S = 0) = 1$ follows immediately from the additional properties of coherent structures $\phi(\underline{1}) = 1$, $\phi(\underline{0}) = 0$.

Theorem 5.4

Let $X_1, X_2, \ldots, X_n$ be independent binary random variables. Let $f(\underline{X})$ be increasing in expectation at $p_1, p_2, \ldots, p_n$. Then

$$\text{cov}[f(\underline{X}), S(\underline{X})] \geq 0$$

at $p_1, p_2, \ldots, p_n$.

Proof

Let $F(k) = E[f(\underline{X})|S = k]$, $k = 0,1,2,\ldots,n$, and $g(\underline{X}) = F[S(\underline{X})]$. Then f increasing in expectation implies g increasing so that $\text{cov}[g,S] \geq 0$ by Theorem 3.1. Since

$$E[g(\underline{X})] = \sum_{k=0}^{n} E[f|S = k] \cdot P[S = k] = E[f]$$

and

$$\begin{aligned} E[g(\underline{X})S(\underline{X})] &= \sum_{k=0}^{n} kE[f|S = k] \cdot P[S = k] \\ &= \sum_{k=0}^{n} E[fS|S = k] \cdot P[S = k] = E[fS], \end{aligned}$$

we have

$$\begin{aligned} \text{cov}[f,S] &= E[fS] - E[f] \cdot E[S] = E[gS] - E[g] \cdot E[S] \\ &= \text{cov}[g,S] \geq 0. \end{aligned}$$

<u>Remark</u>

Under the hypotheses that $X_1, X_2, \ldots, X_n$ are independent binary random variables and $0 < p_i < 1$, $i = 1,2,\ldots,n$, if $\phi(\underline{X})$ is a structure function neither identically zero nor identically one, there is no difficulty in defining the <u>mean path</u> and the <u>mean cut</u> of ϕ by, respectively

$$P(\underline{p}) = E[S|\phi = 1] = \frac{E[S\phi]}{E[\phi]}$$

$$C(\underline{p}) = E[n - S|\phi = 0] = \frac{E[(n - S)(1 - \phi)]}{E(1 - \phi)} .$$

As shown on [1], it is then immediate that

$$P(\underline{p}) + C(\underline{p}) = n + \frac{\text{cov}[\phi, X]}{E[\phi](1 - E[\phi])}$$

In this context the condition $\text{cov}[\phi, S] \geq 0$ is equivalent to $P(\underline{p}) + C(\underline{p}) \geq n$.

<u>Theorem 5.5</u>

Let $X_1, X_2, \ldots, X_n$ be independent binary random variables and $f(\underline{X})$ be a function such that any one of th properties

(a) $\text{cov}[f(\underline{X}), S(\underline{X})] \geq 0$

(b) $\text{cov}[f(\underline{X}), X_i] \geq 0$ for each $i = 1,2,\ldots,n$

(c) f is increasing in expectation

holds for every choice of $p_1, p_2, \ldots, p_n$ for which $0 < p_i < 1$, $i = 1,2,\ldots,n$. Then f is increasing.

<u>Proof</u>

Note that (b) implies (a) since

$$\text{cov}[f(\underline{X}),S(\underline{X})] = \sum_{i=1}^{n} \text{cov}[f(\underline{X}),X_i],$$

and (c) implies (a) by Theorem 5.4. Thus it remains to show that (a) implies that f is increasing.

Now f is increasing if, and only if, f is increasing in each coordinate, i.e., $f(1_i,\underline{x}) \geq f(0_i,\underline{x})$ for all i and $(\cdot_i,\underline{x})$. Suppose f is not increasing. Then there is an i and a vector $(\cdot_i,\underline{x}^0)$ such that $f(1_i,\underline{x}^0) < f(0_i,\underline{x}^0)$. For $j = 1,2,\ldots,n$; $j \neq i$, let

$$p_j \to \begin{cases} 1 & \text{if } x_j^0 = 1 \\ 0 & \text{if } x_j^0 = 0. \end{cases}$$

From Lemma 3.1

$$\text{cov}[f(\underline{X}),X_j] = p_j q_j \cdot E[f(1_j,\underline{X}) - f(0_j,\underline{X})].$$

Since for each $j = 1,2,\ldots,n$

$$E[f(1_j,\underline{X}) - f(0_j,\underline{X})] \leq \alpha_j \equiv \max_{(\cdot_j,\underline{x})} \{f(1_j,\underline{x}) - f(0_j,\underline{x})\} < \infty,$$

we have

$$\text{cov}[f(\underline{X}),X_j] \leq p_j q_j \cdot \alpha_j \to 0, \quad j \neq i.$$

Also

$$E[f(1_i,\underline{X}) - f(0_i,\underline{X})] \leq \{f(1_i,\underline{x}^0) - f(0_i,\underline{x}^0)\}\, P(\cdot_i,\underline{x}^0) + \alpha_i[1 - P(\cdot_i,\underline{x}^0)],$$

where

$$P(\cdot_i, \underline{x}^0) = \prod_{\substack{k=1 \\ k \neq i}}^{n} \Pr[X_j = x_j^0], \quad \text{so that}$$

$$\operatorname{cov}[f(\underline{X}), X_i] \to p_i q_i \cdot \{f(1_i, \underline{x}^0) - f(0_i, \underline{x}^0)\}.$$

Thus, since $p_i q_i > 0$ and $\operatorname{cov}[f, S] \geq 0$ during the limiting process, we show $f(1_i, \underline{x}^0) \geq f(0_i, \underline{x}^0)$ and nave a contradiction.

6. The multivariate Moore-Shannon inequality

The univariate Moore-Shannon inequality

$$(6.1) \qquad p(1-p)\frac{dh(p)}{dp} \geq h(p)[1-h(p)]$$

compares structural reliability $h(p)$ with component reliability p in the case of semi-coherent systems whose components are of identical reliability. The inequality is strict except when $h(p) \equiv 0$, $h(p) \equiv 1$, or $h(p) \equiv p$. It is derived for two-terminal networks in [3] and for coherent structures in [1]. The inequality is the principal tool in the demonstration of the S-shaped relationship between structural reliability and component reliability. We shall obtain a multivariate generalization, directly from Theorem 3.1, which permits a much simplified proof of the S-shapedness result.

We will need

Theorem 6.1

Let $X_1, X_2, \ldots, X_n$ be independent binary random variables such that $0 < p_i < 1$, $i = 1,2,\ldots,n$. Let $\phi(\underline{x})$ be the structure function of a coherent structure having at least two essential components. Then

$$\mathrm{cov}[\phi(\underline{X}), S(\underline{X}) - \phi(\underline{X})] > 0,$$

where $S(\underline{x}) = \sum_{i=1}^{n} x_i$.

Proof

To apply Theorem 3.1 and its corollary we must show first that $S - \phi$ is increasing and second that ϕ and $S - \phi$ have an essential x_i in common.

To show $S - \phi$ increasing is to show that for each vector $(\cdot_i, \underline{x})$

$$S(1_i, \underline{x}) - \phi(1_i, \underline{x}) \geq S(0_i, \underline{x}) - \phi(0_i, \underline{x})$$

which is equivalent to

$$1 \geq \phi(1_i, \underline{x}) - \phi(0_i, \underline{x}),$$

a statement which follows immediately from the properties of ϕ.

To show that ϕ and $S - \phi$ have an essential x_i in common suppose that x_i, x_j are essential to ϕ, and that neither x_i or x_j is essential to $S - \phi$. Then

$$S(1_i, \underline{x}) - \phi(1_i, \underline{x}) = S(0_i, \underline{x}) = \phi(0_i, \underline{x}), \quad \text{all} \quad (\cdot_i, \underline{x}).$$

$$S(1_j, \underline{x}) - \phi(1_j, \underline{x}) = S(0_j, \underline{x}) - \phi(0_j, \underline{x}), \quad \text{all} \quad (\cdot_j, \underline{x}).$$

Thus

$$\phi(1_i, \underline{x}) - \phi(0_i, \underline{x}) = 1, \quad \text{all} \quad (\cdot_i, \underline{x}),$$

$$\phi(1_j, \underline{x}) - \phi(0_j, \underline{x}) = 1, \quad \text{all} \quad (\cdot_j, \underline{x}),$$

which implies

$$\phi(1_i, \underline{x}) \equiv 1 \qquad \phi(0_i, \underline{x}) \equiv 0$$

$$\phi(1_j, \underline{x}) \equiv 1 \qquad \phi(0_j, \underline{x}) \equiv 0.$$

But a contradiction arises from the consideration of $\phi(1_i, 0_j, \underline{x})$ or $\phi(0_i, 1_j, \underline{x})$.

We can now prove the desired generalization of the Moore-Shannon inequality:

Theorem 6.2

Let $X_1, X_2, \ldots, X_n$ be independent binary random variables such that $0 < p_i < 1$, $i = 1,2,\ldots,n$. Let $\phi(\underline{x})$ be the structure function of a coherent structure such that $\phi(\underline{x}) \not\equiv x_i$ for $i = 1,2,\ldots,n$. Then

$$\sum_{i=1}^{n} p_i(1 - p_i) \frac{\partial h(\underline{p})}{\partial p_i} > h(\underline{p})[1 - h(\underline{p})],$$

where $h(\underline{p}) = E[\phi(\underline{X}) | \underline{p}]$.

Proof

By hypothesis, ϕ has at least two essential components. The inequality of Theorem 6.1 can be rewritten as

$$(6.2) \qquad \sum_{i=1}^{n} \operatorname{cov}[\phi, X_i] = \operatorname{cov}[\phi, S] > E[\phi]\{1 - E[\phi]\}.$$

Since $h = E[\phi]$ and from (5.3)

$$(6.3) \qquad \operatorname{cov}[\phi, X_i] = p_i(1 - p_i) \frac{\partial h}{\partial p_i} ,$$

the result follows.

We refer to the result of Theorem 6.2 as the multivariate Moore-Shannon inequality. It is clearly valid, but without strict inequality, when $\phi \equiv 0$, $\phi \equiv 1$, or $\phi \equiv x_i$.

Application to S-shapedness

Let $\phi(\underline{x})$ be a coherent structure function and $h(\underline{p})$ be the corresponding structural reliability function. Assume that all components have identical reliability, i.e., $p_i = p$, $i = 1,2,\ldots,n$. Then

$$\frac{dh}{dp} = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i} \frac{dp_i}{dp} = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i}$$

so that by Theorem 6.2

$$p(1 - p)\frac{dh}{dp} = \sum_{i=1}^{n} p_i(1 - p_i)\frac{\partial h}{\partial p_i} > h(1 - h), \quad 0 < p < 1,$$

unless $\phi \equiv x_i$ for some i. Thus except for the three exceptional cases if $h(p_0) = p_0$, $0 < p_0 < 1$, then $\frac{dh}{dp} > 1$ at p_0. It follows that the curve $h(p)$ can cross the diagonal p only from below, if it crosses at all.

From (5.2)

$$\left.\frac{\partial h}{dp_i}\right|_{p=0} = \phi(1_i,\underline{0}) - \phi(0_i,\underline{0}) = \begin{cases} 1 & \text{if } (1_i,\underline{0}) \text{ is a path of the structure} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\left.\frac{\partial h}{\partial p_i}\right|_{p=1} = \phi(1_i,\underline{1}) - \phi(0_i,\underline{1}) = \begin{cases} 1 & \text{if } (0_i,\underline{1}) \text{ is a cut of the structure} \\ 0 & \text{otherwise} \end{cases}$$

It follows that if no minimal path set or minimal cut set of the structure consists of just one component, $\frac{dh}{dp} = 0$ at $p = 0$ and $p = 1$. Then $h(p)$ must cross the diagonal p at least once (from below) and can cross only once.

If some one component is a "minimal path", then clearly the structure is more reliable than that component, i.e., $h(p) \geq p$, $0 \leq p \leq 1$. If some one component is a "minimal cut" then the structure is less reliable than that component, i.e., $h(p) \leq p$, $0 \leq p \leq 1$ (see [3], [1]). Thus, in these cases $h(p)$ does not cross the diagonal at all.

Quite often one wants to study the changing relationship between structural reliability and component reliablities when component reliabilities vary as a function of some parameter. One case of this is the S-shapedness relationship just considered, where $p_i(p) = p$, $i = 1,2,\ldots,n$, $0 \leq p \leq 1$. Another is the common situation in which each component reliability is supposed to be a decreasing function $p_i(t)$ of the accumulated operating time t of the system, $0 \leq t < \infty$. In problems of reliability growth one might want to treat each component reliability as an increasing function of a parameter representing development effort or expenditure. The following theorem gives one way of writing the Moore-Shannon inequality in such situations.

<u>Theorem 6.3</u>

Let $X_i, X_2, \ldots, X_n$ be independent binary random variables. Let $\phi(\underline{x})$ be a coherent structure function such that $\phi(\underline{x}) \not\equiv x_i$ for $i = 1,2,\ldots,n$. Suppose for $-\infty \leq a < \mu < b \leq \infty$, $p_i(\mu) = P[X_i = 1|\mu]$ is such that $\frac{dp_i}{d\mu}$ exists, $i = 1,2,\ldots,n$. Then $\frac{dh}{d\mu}$ exists, $a < \mu < b$, and for $a < \mu < b$, $0 < p_i(\mu) < 1$, $i = 1,2,\ldots n$,

$$\frac{dh/d\mu}{h(1-h)} > \sum_{i=1}^{n} c_i(\mu) \frac{dp_i/d\mu}{p_i(1-p_i)},$$

where $c_i(\mu) = \dfrac{\operatorname{cov}[\phi(\underline{X}), X_i|\mu]}{\operatorname{cov}[\phi(\underline{X}), S(\underline{X})|\mu]}$ and $\sum_{i=1}^{n} c_i(\mu) = 1.$

Proof

Write

$$\frac{dh}{d\mu} = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i} \cdot \frac{dp_i}{d\mu} = \sum_{i=1}^{n} p_i q_i \frac{\partial h}{\partial p_i} \cdot \frac{dp_i/d\mu}{p_i q_i} .$$

Thus, using (5.3),

$$\frac{dh}{d\mu} = \sum_{i=1}^{n} \mathrm{cov}[\phi, X_i] \cdot \frac{dp_i/d\mu}{p_i q_i} .$$

Dividing by, respectively, the two sides of (6.2) gives

$$\frac{dh/d\mu}{h(1-h)} > \sum_{i=1}^{n} \frac{\mathrm{cov}[\phi, X_i]}{\mathrm{cov}[\phi, S]} \cdot \frac{dp_i/d\mu}{p_i q_i} .$$

We use Theorem 6.3 to obtain S-shapedness results in the case of components of differing reliabilities. Specifically:

Theorem 6.4

Let $p_i(\mu)$ satisfy for $i = 1,2,\ldots,n$

(6.3) $$\frac{p'_i(\mu)}{p_i(\mu)[1 - p_i(\mu)]} \geq \frac{1}{\mu(1-\mu)} , \quad 0 < \mu < 1$$

and let $h[\underline{p}(\mu)]$ be the reliability function of a coherent structure such that $\phi(\underline{x}) \not\equiv x_i$ for $i = 1,2,\ldots,n$. Then

(6.4) $$\frac{dh/d\mu}{h(1-h)} > \frac{1}{\mu(1-\mu)} , \quad 0 < \mu < 1.$$

(6.4) implies that $h[\underline{p}(\mu)]$ crosses the diagonal at most once.

Proof

For $0 < \mu < 1$,

$$\frac{dh/d\mu}{h(1-h)} > \sum_{i=1}^{n} c_i(\mu) \frac{p_i'(\mu)}{p_i(1-p_i)} \geq \frac{1}{\mu(1-\mu)} \sum_{i=1}^{n} c_i(\mu) = \frac{1}{\mu(1-\mu)} ;$$

the first inequality follows from Theorem 6.3, the second by hypothesis. The final equality is a consequence of the fact that $\sum_{i=1}^{n} c_i(\mu) = 1$, the $c_i(\mu)$ being defined in Theorem 6.3.

We can go a step further; we can state conditions under which $h[\underline{p}(\mu)]$ actually does cross the diagonal Θ precisely once.

Corollary

Assume $p_i(\mu)$ satisfies (6.3) and that $p_i(\mu)$ actually crosses the diagonal once for $i = 1,2,\ldots,n$. Assume further that $h(p)$ is the reliability function of a coherent structure such that $\phi(\underline{x}) \not\equiv x_i$ for $i = 1,2,\ldots,n$ and that $h(p)$ actually crosses the diagonal exactly once. Then $h[\underline{p}(\mu)]$ crosses the diagonal exactly once.

Proof

For μ sufficiently close to 0, each $p_i(\mu) < \mu$, so that

$h[\underline{p}(\mu)] < h(\underline{\mu}) < \mu$.

Similarly for μ sufficiently close to 1, each $p_i(\mu) > \mu$, so that

$h[\underline{p}(\mu)] > h(\underline{\mu}) > \mu$.

Thus $h[\underline{p}(\mu)]$ crosses the diagonal at least once. Since we know from Theorem 6.4 that $h[\underline{p}(\mu)]$ crosses the diagonal at most once, the conclusion follows.

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